

# REGULARITY AND CAPACITY FOR THE FRACTIONAL DISSIPATIVE OPERATOR

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**ABSTRACT.** This note is devoted to exploring some analytic-geometric properties of the regularity and capacity associated to the so-called fractional dissipative operator  $\partial_t + (-\Delta)^\alpha$ , naturally establishing a diagonally sharp Hausdorff dimension estimate for the blow-up set of a weak solution to the fractional dissipative equation  $(\partial_t + (-\Delta)^\alpha)u(t, x) = F(t, x)$  subject to  $u(0, x) = 0$ .

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## 1. INTRODUCTION AND THE MAIN RESULTS

This beginning part is designed to describe the principal results of this article.

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**1.1. The fractional dissipative equation.** For  $n = 1, 2, 3, \dots$ ,  $\alpha \in (0, 1]$  and  $\mathbb{R}_+ := (0, \infty)$ , let  $\mathbb{R}_+^{1+n} := \mathbb{R}_+ \times \mathbb{R}^n$  be the upper half space of the  $1 + n$  dimensional Euclidean space  $\mathbb{R}^{1+n}$  and  $(-\Delta)^\alpha$  be the fractional Laplace operator which is determined by

$$(-\Delta)^\alpha u(\cdot, x) := \mathcal{F}^{-1}(|\xi|^{2\alpha} \mathcal{F}u(\cdot, \xi))(x), \quad \forall x \in \mathbb{R}^n,$$

where  $\mathcal{F}$  denotes the Fourier transform and  $\mathcal{F}^{-1}$  its inverse:

$$\begin{cases} \mathcal{F}(g)(x) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix \cdot y} g(y) dy; \\ \mathcal{F}^{-1}(g)(x) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix \cdot y} g(y) dy. \end{cases}$$

From the celebrated Duhamel's principle it follows that a weak solution  $u(t, x)$  to the fractional dissipative equation living in fluid dynamics via the so-called fractional dissipative operator  $L^{(\alpha)} := \partial_t + (-\Delta)^\alpha$ :

$$\begin{cases} L^{(\alpha)} u(t, x) = F(t, x), & \forall (t, x) \in \mathbb{R}_+^{1+n}; \\ u(0, x) = f(x), & \forall x \in \mathbb{R}^n, \end{cases}$$

namely (cf. [9, 7]),

$$\begin{cases} \iint_{\mathbb{R}_+^{1+n}} u \tilde{L}^{(\alpha)} \phi \, dx dt = - \iint_{\mathbb{R}_+^{1+n}} F \phi \, dx dt - \int_{\mathbb{R}^n} f(x) \phi(0, x) \, dx, \quad \forall \phi \in C_0^\infty(\mathbb{R}_+^{1+n}) \\ \text{with} \\ \tilde{L}^{(\alpha)} \phi(t, x) = -\partial_t \phi(t, x) + \left( \frac{(1-\alpha)2^{2\alpha}\Gamma(\frac{n+2\alpha}{2})}{\pi^{n/2}\Gamma(1-\alpha)} \right) \lim_{\epsilon \rightarrow 0} \int_{\{y \in \mathbb{R}^n: |y| > \epsilon\}} \frac{\phi(t, x+y) - \phi(t, x)}{|y|^{-n-2\alpha}} dy, \end{cases}$$

can be written as

$$u(t, x) = R_\alpha f(t, x) + S_\alpha F(t, x),$$

where

$$\begin{cases} R_\alpha f(t, x) := e^{-t(-\Delta)^\alpha} f(x); \\ S_\alpha F(t, x) := \int_0^t e^{-(t-s)(-\Delta)^\alpha} F(s, x) \, ds, \end{cases}$$

for which

$$\begin{cases} e^{-t(-\Delta)^\alpha} v(\cdot, x) := K_t^{(\alpha)}(x) * v(\cdot, x); \\ K_t^{(\alpha)}(x) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix \cdot y - t|y|^{2\alpha}} dy, \end{cases}$$

and  $*$  represents the convolution operating on the space variable. Here it is perhaps appropriate to mention that

$$\begin{cases} K_t^{(1)}(x) = (4\pi)^{-n/2} e^{-|x|^2/(4t)} \\ \text{and} \\ K_t^{(1/2)}(x) = \pi^{-(1+n)/2} \Gamma((n+1)/2) t(t^2 + |x|^2)^{-(1+n)/2} \end{cases}$$

are the heat and Poisson kernels, respectively. Of course,  $\Gamma(\cdot)$  is the classical gamma function. Although an explicit formula of  $K_t^{(\alpha)}(x)$  for  $\alpha \in (0, 1] \setminus \{1/2, 1\}$  is unknown (cf. [8, 6, 15, 14, 4] and [19, 9, 10, 11, 12] for some

related information), one has the following basic estimate (cf. [17, 5]): under  $\alpha \in (0, 1)$

$$K_t^{(\alpha)}(x) \approx t(t^{\frac{1}{2\alpha}} + |x|)^{-(n+2\alpha)}, \quad \forall (t, x) \in \mathbb{R}_+^{1+n}.$$

In the above and below,  $X \approx Y$  means  $Y \lesssim X \lesssim Y$  where the second estimate means that there is a positive constant  $c$ , independent of main parameters, such that  $X \leq cY$ . From now on,  $\alpha$  will be always assumed to be in the interval  $(0, 1)$ .

**1.2. Regularity for the fractional dissipative operator.** The following function space regularity results of Strichartz type, plus [1], actually induce the research objective of this current paper.

**Theorem 1.1.**

(i) [8, Lemma 3.2] *If*

$$\begin{cases} 1 \leq p \leq \tilde{p} < \frac{np}{n-\min\{n, 2\alpha\}}; \\ \frac{1}{\tilde{q}} = (\frac{n}{2\alpha})(\frac{1}{p} - \frac{1}{\tilde{p}}), \end{cases}$$

*then*

$$\|R_\alpha f\|_{L_t^{\tilde{q}} L_x^{\tilde{p}}(\mathbb{R}_+^{1+n})} \lesssim \|f\|_{L^p(\mathbb{R}^n)}.$$

(ii) [18, Theorem 1.4] *If*

$$\begin{cases} 1 \leq p < \tilde{p} \leq \infty; \\ 1 < q < \tilde{q} < \infty; \\ (\frac{1}{q} - \frac{1}{\tilde{q}}) + (\frac{n}{2\alpha})(\frac{1}{p} - \frac{1}{\tilde{p}}) = 1, \end{cases}$$

*then*

$$\|S_\alpha F\|_{L_t^{\tilde{q}} L_x^{\tilde{p}}(\mathbb{R}_+^{1+n})} \lesssim \|F\|_{L_t^q L_x^p(\mathbb{R}_+^{1+n})}.$$

Here and henceforth:  $L_x^p(\mathbb{R}^n)$  denotes the usual Lebesgue  $1 \leq p \leq \infty$ -space with respect to the space variable  $x$ ;  $L_t^{p_2} L_x^{p_1}(\mathbb{R}_+^{1+n})$  is the mixed  $(1 \leq p_1, p_2 < \infty)$ -Lebesgue space of all functions  $F$  on  $\mathbb{R}_+^{1+n}$  with

$$\|F\|_{L_t^{p_2} L_x^{p_1}(\mathbb{R}_+^{1+n})} := \left( \int_{\mathbb{R}_+} \left[ \int_{\mathbb{R}^n} |F(t, x)|^{p_1} dx \right]^{\frac{p_2}{p_1}} dt \right)^{\frac{1}{p_2}} < \infty,$$

where a suitable modification is needed whenever  $p_1$  or  $p_2$  is  $\infty$ ; for  $\mathbb{X} = \mathbb{R}^n$  or  $\mathbb{R}_+^{1+n}$  the symbols  $C^\infty(\mathbb{X})$ ,  $C_0^\infty(\mathbb{X})$  and  $C(\mathbb{X})$  stand for all infinitely smooth functions in  $\mathbb{X}$ , all infinitely smooth functions with compact support in  $\mathbb{X}$  and all continuous functions in  $\mathbb{X}$ , respectively.

Throughout the paper, for each  $(t_0, x_0) \in \mathbb{R}_+^{1+n}$  and  $r > 0$ , the parabolic ball is defined as

$$B_r^{(\alpha)}(t_0, x_0) := \{(t, x) \in \mathbb{R}_+^{1+n} : |t - t_0| < r^{2\alpha} \text{ \& } |x - x_0| < r\}$$

and its volume is denoted by  $|B_{r_0}^{(\alpha)}(t_0, x_0)| \approx r_0^{n+2\alpha}$ .

The first main result of this paper appears as an essential extension or complement of Theorem 1.1.

**Theorem 1.2.**

- (i) If  $p \in [1, \infty]$  and  $f \in L^p(\mathbb{R}^n)$ , then  $R_\alpha f$  is continuous on  $\mathbb{R}_+^{1+n}$ .  
(ii) If

$$\begin{cases} p \in [1, \infty); \\ 1 < q < \infty; \\ \frac{n}{p} + \frac{2\alpha}{q} = 2\alpha; \\ (t_0, x_0) \in \mathbb{R}_+^{1+n}; \\ r_0 = t_0^{\frac{1}{2\alpha}}; \\ 0 < \|F\|_{L_t^q L_x^p(\mathbb{R}_+^{1+n})} < \infty, \end{cases}$$

then there exists  $C > 0$  such that

$$\frac{1}{|B_{r_0}^{(\alpha)}(t_0, x_0)|} \iint_{B_{r_0}^{(\alpha)}(t_0, x_0)} \exp\left(\frac{S_\alpha F(t, x)}{C\|F\|_{L_t^q L_x^p(\mathbb{R}_+^{1+n})}}\right)^{\frac{q}{q-1}} dx dt \lesssim 1.$$

- (iii) If

$$\begin{cases} p \in [1, \infty); \\ 1 < q < \infty; \\ \frac{n}{p} + \frac{2\alpha}{q} < 2\alpha; \\ (t, x) \in \mathbb{R}_+^{1+n}; \\ \|F\|_{L_t^q L_x^p(\mathbb{R}_+^{1+n})} < \infty, \end{cases}$$

then  $S_\alpha F$  is Hölder continuous in the sense that

$$|S_\alpha F(t, x) - S_\alpha F(t_0, x_0)| \lesssim (|t - t_0|^{\frac{2\alpha - \frac{n}{p} - \frac{2\alpha}{q}}{2\alpha}} + |x - x_0|^{2\alpha - \frac{n}{p} - \frac{2\alpha}{q}}) \|F\|_{L_t^q L_x^p(\mathbb{R}_+^{1+n})}$$

holds for any two sufficient close points  $(t_0, x_0), (t, x) \in \mathbb{R}_+^{1+n}$

**1.3. Capacity for the fractional dissipative operator.** From Theorems 1.1-1.2 we know that it is necessary to estimate the size of the blow-up set of the so-called fractional dissipative potential  $S_\alpha F$  below:

$$\mathcal{B}[S_\alpha F; p, q] := \{(t, x) \in \mathbb{R}_+^{1+n} : S_\alpha F(t, x) = \infty\} \quad \text{for } 0 \leq F \in L_t^q L_x^p(\mathbb{R}_+^{1+n}).$$

To handle this issue, let us introduce a new type of capacity. For a compact subset  $K$  of  $\mathbb{R}_+^{1+n}$ , let

$$C_{p,q}^{(\alpha)}(K) := \inf \left\{ \|F\|_{L_t^q L_x^p(\mathbb{R}_+^{1+n})}^{p \wedge q} : F \geq 0 \text{ \& } S_\alpha F \geq 1_K \right\}$$

be the  $(\alpha, p, q)$ -capacity of  $K$  for the fractional dissipative operator  $L^{(\alpha)}$ , where  $1_K$  is the characteristic function of  $K$ ,  $p \wedge q := \min\{p, q\}$ , and  $1 \leq$

$p, q < \infty$ . Moreover, the definition of  $C_{p,q}^{(\alpha)}$  extends to any subset of  $\mathbb{R}_+^{1+n}$  in a similar way as [2, Definitions 2.2.2 & 2.2.4].

Next, for

$$\begin{cases} 0 < \varepsilon \leq \infty; \\ 0 < d < \infty; \\ K \subset \mathbb{R}_+^{1+n}; \\ B_{r_j}^{(\alpha)}(t_j, x_j) := \{(s, y) \in \mathbb{R}_+^{1+n} : |s - t_j| < r_j^{2\alpha} \text{ \& } |y - x_j| < r_j\}; \\ (t_j, x_j, r_j) \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}_+; \\ \phi : [0, \infty) \mapsto [0, \infty] \text{ -- an increasing function with } \phi(0) = 0, \end{cases}$$

let

$$H_\varepsilon^{\phi, \alpha}(K) := \inf \left\{ \sum_{j=1}^{\infty} \phi(r_j) : K \subseteq \bigcup_{j=1}^{\infty} B_{r_j}^{(\alpha)}(t_j, x_j); \quad \text{with } r_j \in (0, \varepsilon) \right\}$$

be the  $L^\alpha$ -based  $(\phi, \varepsilon)$ -Hausdorff capacity of  $K$ . Then the  $L^\alpha$ -based  $\phi$ -Hausdorff measure of  $K$  is defined by

$$H^{\phi, \alpha}(K) := \lim_{\varepsilon \rightarrow 0} H_\varepsilon^{\phi, \alpha}(K).$$

If  $\phi(r) := r^d$  for all  $r \in (0, \infty)$ , then

$$\begin{cases} H_\varepsilon^{\phi, \alpha}(K) \equiv H_\varepsilon^{d, \alpha}(K); \\ H^{\phi, \alpha}(K) \equiv H^{d, \alpha}(K); \\ \dim_H^{(\alpha)}(K) := \inf\{d : H^{d, \alpha}(K) = 0\}, \end{cases}$$

where the last quantity is called the  $L^{(\alpha)}$ -based Hausdorff dimension of  $K$ .

Below is our second theorem.

**Theorem 1.3.**

(i) If

$$\begin{cases} 1 \leq p < \infty; \\ 1 < q < \infty; \\ \frac{n}{p} + \frac{2\alpha}{q} - 2\alpha > 0, \end{cases}$$

then

$$C_{p,q}^{(\alpha)}(B_{r_0}^{(\alpha)}(t_0, x_0)) \approx r_0^{(p \wedge q)(\frac{n}{p} + \frac{2\alpha}{q} - 2\alpha)} \quad \text{as } r_0 \rightarrow 0 \text{ \& } (r_0, x_0) \in \mathbb{R}_+^{1+n}.$$

(ii) If

$$\begin{cases} 1 \leq p < \infty; \\ 1 < q < \infty; \\ \frac{n}{p} + \frac{2\alpha}{q} - 2\alpha = 0, \end{cases}$$

then

$$C_{p,q}^{(\alpha)}(B_r^{(\alpha)}(t_0, x_0)) \approx \left( \ln \frac{1}{r_0} \right)^{(p \wedge q)(\frac{1}{q}-1)} \quad \text{as } r_0 \rightarrow 0 \text{ \& } (r_0, x_0) \in \mathbb{R}_+^{1+n}.$$

As an immediate consequence of Theorems 1.1-1.2-1.3, we get not only three geometric inequalities linking two types of capacity, but also some Hausdorff dimension estimates for the blow-up sets which are sharp in the diagonal case  $p = q$ .

**Corollary 1.4.**

(i) Let  $\mathcal{L}^1(A)$  and  $\mathcal{L}^n(B)$  stand for the 1-dimensional and  $n$ -dimensional Lebesgue measures of bounded Borel sets  $A \subset \mathbb{R}_+$  and  $B \subset \mathbb{R}^n$ , respectively. If

$$\begin{cases} 1 \leq p < \tilde{p} < \infty; \\ 1 < q < \tilde{q} < \infty; \\ \beta := (p \wedge q)(\frac{n}{p} + \frac{2\alpha}{q} - 2\alpha) > 0; \\ (\frac{1}{q} - \frac{1}{\tilde{q}}) + (\frac{n}{2\alpha})(\frac{1}{p} - \frac{1}{\tilde{p}}) = 1, \end{cases}$$

then there is a  $\delta \in (0, 1)$  such that

$$(\mathcal{L}^1(A))^{\frac{p \wedge q}{q}} (\mathcal{L}^n(B))^{\frac{p \wedge q}{p}} \lesssim C_{p,q}^{(\alpha)}(A \times B) \lesssim H_\delta^{\beta, \alpha}(A \times B).$$

(ii) Let  $K$  be a compact subset of  $\mathbb{R}_+^{1+n}$ . If

$$\begin{cases} 1 \leq p < \infty; \\ 1 < q < \infty; \\ \beta := (p \wedge q)(\frac{n}{p} + \frac{2\alpha}{q} - 2\alpha) > 0, \end{cases}$$

then there is a  $\delta \in (0, 1)$  such that

$$C_{p,q}^{(\alpha)}(K) \lesssim H_\delta^{\beta, \alpha}(K),$$

and hence

$$\dim_H^{(\alpha)}(\mathcal{B}[S_\alpha F; p, q]) \leq n - 2\alpha(p \wedge q - 1) \text{ provided } n - 2\alpha(p \wedge q - 1) > 0.$$

(iii) Let  $K$  be a compact subset of  $\mathbb{R}_+^{1+n}$ . If

$$\begin{cases} 1 \leq p < \infty; \\ 1 < q < \infty; \\ \frac{n}{p} + \frac{2\alpha}{q} - 2\alpha = 0; \\ \phi(r) := (\ln_+ \frac{1}{r})^{-(p \wedge q)(1-\frac{1}{q})}, \quad \forall r \in \mathbb{R}_+; \\ \ln_+ t := \max\{0, \ln t\}, \quad \forall t \in \mathbb{R}_+, \end{cases}$$

then there is a  $\delta \in (0, 1)$  such that

$$C_{p,q}^{(\alpha)}(K) \lesssim H_\delta^\phi(K),$$

and hence

$$H^{\phi_{\epsilon}, \alpha}(\mathcal{B}[S_{\alpha}F; p, q]) = 0 \text{ provided } \begin{cases} n - 2\alpha(p \wedge q - 1) = 0; \\ \phi_{\epsilon}(r) := (\ln_+ \frac{1}{r})^{-p \wedge q - \epsilon}, \quad \forall r \in \mathbb{R}_+; \\ \epsilon > 0. \end{cases}$$

## 2. BASICS OF THE $(\alpha, p, q)$ -CAPACITY

In order to demonstrate Theorems 1.2-1.3 and Corollary 1.4, we need to know some basic facts on the  $(\alpha, p, q)$ -capacity.

**2.1. Duality of the  $(\alpha, p, q)$ -capacity.** To establish the adjoint formulation of  $C_{p,q}^{(\alpha)}$ , we need to find out adjoint operator  $S_{\alpha}^*$  of  $S_{\alpha}$ . Note that for any  $F, G \in C_0^{\infty}(\mathbb{R}_+^{1+n})$  one has

$$\iint_{\mathbb{R}_+^{1+n}} S_{\alpha}F(t, x)G(t, x) dx dt = \int_{\mathbb{R}_+^{1+n}} F(t, x) \left( \int_t^{\infty} e^{-(s-t)(-\Delta)^{\alpha}} G(s, x) ds \right) dx dt.$$

Thus,  $S_{\alpha}^*G$  is given by setting, for all  $(t, x) \in \mathbb{R}_+^{1+n}$ ,

$$(S_{\alpha}^*G)(t, x) := \int_t^{\infty} e^{-(s-t)(-\Delta)^{\alpha}} G(s, x) ds, \quad \forall G \in C_0^{\infty}(\mathbb{R}_+^{1+n}).$$

The definition of  $S_{\alpha}^*$  can be extended to the family of Borel measures  $\mu$  with compact support in  $\mathbb{R}_+^{1+n}$ . In fact, note that if  $F$  is continuous and has a compact support in  $\mathbb{R}_+^{1+n}$  and  $\|\mu\|_1$  stands for the total variation of  $\mu$  then a simple calculation with the equivalent estimate

$$K_t^{(\alpha)}(x) \approx t(t^{1/(2\alpha)} + |x|)^{-n-2\alpha}, \quad \forall (t, x) \in \mathbb{R}_+^{1+n},$$

gives

$$\left| \iint_{\mathbb{R}_+^{1+n}} S_{\alpha}F d\mu \right| \lesssim \|\mu\|_1 \sup_{(t,x) \in \mathbb{R}_+^{1+n}} |F(t, x)|.$$

Hence an application of the Riesz representation theorem yields a Borel measure  $\nu$  on  $\mathbb{R}_+^{1+n}$  such that

$$\iint_{\mathbb{R}_+^{1+n}} S_{\alpha}F d\mu = \int_{\mathbb{R}_+^{1+n}} F d\nu.$$

This indicates that  $S_{\alpha}^*\mu$  may be defined by  $\nu$ .

The above analysis leads to a dual description of the  $(\alpha, p, q)$ -capacity.

**Proposition 2.1.** *For a compact subset  $K$  of  $\mathbb{R}_+^{1+n}$  let  $\mathcal{M}_+(K)$  be the class of all positive measures on  $\mathbb{R}_+^{1+n}$  supported by  $K$ . If*

$$\begin{cases} 1 < p, q < \infty; \\ p' = p/(p-1); \\ q' = q/(q-1), \end{cases}$$

then

$$C_{p,q}^{(\alpha)}(K) = \sup \{ \|\mu\|_1^{p \wedge q} : \mu \in \mathcal{M}_+(K) \text{ \& } \|S_\alpha^* \mu\|_{L_t^{q'} L_x^{p'}(\mathbb{R}_+^{1+n})} \leq 1 \} =: \tilde{C}_{p,q}^{(\alpha)}(K).$$

*Proof.* Since

$$\begin{aligned} \|\mu\|_1 &= \mu(K) \\ &\leq \iint_{\mathbb{R}_+^{1+n}} S_\alpha F d\mu \\ &= \iint_{\mathbb{R}_+^{1+n}} F S_\alpha^* \mu dx dt \\ &\leq \|F\|_{L_t^q L_x^p(\mathbb{R}_+^{1+n})} \|S_\alpha^* \mu\|_{L_t^{q'} L_x^{p'}(\mathbb{R}_+^{1+n})}, \end{aligned}$$

one has

$$\tilde{C}_{p,q}^{(\alpha)}(K) \leq C_{p,q}^{(\alpha)}(K)$$

for any compact set  $K \subset \mathbb{R}_+^{1+n}$ . Moreover, this last inequality is actually an equality - in fact, if

$$\begin{cases} X = \{\mu : \mu \in \mathcal{M}_+(K) \text{ \& } \mu(K) = 1\}; \\ Y = \{F : 0 \leq F \in L_t^q L_x^p(\mathbb{R}_+^{1+n}) \text{ \& } \|F\|_{L_t^q L_x^p(\mathbb{R}_+^{1+n})} \leq 1\}; \\ Z = \{F : 0 \leq F \in L_t^q L_x^p(\mathbb{R}_+^{1+n}) \text{ \& } S_\alpha F \geq 1_K\}; \\ E(\mu, F) = \iint_{\mathbb{R}_+^{1+n}} (S_\alpha^* \mu) F dx dt = \iint_{\mathbb{R}_+^{1+n}} S_\alpha F d\mu, \end{cases}$$

then an easy computation, along with an application of [2, Theorem 2.4.1], gives

$$\begin{aligned} \min_{\mu \in \mathcal{M}_+(K)} \frac{\|S_\alpha^* \mu\|_{L_t^{q'} L_x^{p'}(\mathbb{R}_+^{1+n})}}{\mu(K)} &= \min_{\mu \in X} \sup_{F \in Y} E(\mu, F) \\ &= \sup_{F \in Y} \min_{\mu \in X} E(\mu, F) \\ &= \sup_{0 \leq F \in L_t^q L_x^p(\mathbb{R}_+^{1+n})} \frac{\min_{(t,x) \in K} S_\alpha F(t, x)}{\|F\|_{L_t^q L_x^p(\mathbb{R}_+^{1+n})}} \\ &= \sup_{F \in Z} \|F\|_{L_t^q L_x^p(\mathbb{R}_+^{1+n})}^{-1} \\ &= (C_{p,q}^{(\alpha)}(K))^{-\frac{1}{p \wedge q}}, \end{aligned}$$

and hence

$$\tilde{C}_{p,q}^{(\alpha)}(K) \geq C_{p,q}^{(\alpha)}(K),$$

thereby the desired equality follows.  $\square$



**2.2. Essentialness of the  $(\alpha, p, q)$ -capacity.** Some fundamental properties of the  $(\alpha, p, q)$ -capacity are stated in the following proposition.

**Proposition 2.2.**

(i)  $C_{p,q}^{(\alpha)}(\emptyset) = 0$ . Moreover, under  $\emptyset \neq K \subset \mathbb{R}_+^{1+n}$ ,  $C_{p,q}^{(\alpha)}(K) = 0$  if and only if there exists  $0 \leq F \in L_t^q L_x^p(\mathbb{R}_+^{1+n})$  such that

$$K \subseteq \{(t, x) \in \mathbb{R}_+^{1+n} : S_\alpha F(t, x) = \infty\}.$$

(ii)  $K_1 \subseteq K_2 \subset \mathbb{R}_+^{1+n} \implies C_{p,q}^{(\alpha)}(K_1) \leq C_{p,q}^{(\alpha)}(K_2)$ .

(iii)

$$C_{p,q}^{(\alpha)}\left(\bigcup_{j=1}^{\infty} K_j\right) \leq \sum_{j=1}^{\infty} C_{p,q}^{(\alpha)}(K_j)$$

for any sequence  $\{K_j\}_{j=1}^{\infty}$  of subsets of  $\mathbb{R}_+^{n+1}$ .

(iv)  $C_{p,q}^{(\alpha)}(K + (0, x_0)) = C_{p,q}^{(\alpha)}(K)$  for any  $K \subset \mathbb{R}_+^{n+1}$  and any  $x_0 \in \mathbb{R}^n$ .

*Proof.* (i) Only the ‘iff’ part needs an argument. To do so, note that for  $\lambda > 0$  the inequality

$$C_{p,q}^{(\alpha)}(\{(t, x) \in \mathbb{R}_+^{1+n} : F \geq 0 \text{ \& } S_\alpha F(t, x) \geq \lambda\}) \leq \lambda^{-p \wedge q} \|F\|_{L_t^q L_x^p(\mathbb{R}_+^{1+n})}^{p \wedge q}$$

follows from the definition of  $C_{p,q}^{(\alpha)}$ . Clearly, this implies

$$C_{p,q}^{(\alpha)}(\mathcal{B}[S_\alpha F; p, q]) = 0.$$

Therefore, if  $0 \leq F \in L_t^q L_x^p(\mathbb{R}_+^{1+n})$  enjoys  $K \subseteq \mathcal{B}[S_\alpha F; p, q]$ , then  $C_{p,q}^{(\alpha)}(K) = 0$  follows from (ii) - the monotonicity of capacity.

Conversely, if  $C_{p,q}^{(\alpha)}(K) = 0$  then taking nonnegative functions  $F_j$  such that

$$\begin{cases} S_\alpha F_j(t, x) \geq 1, \quad \forall (t, x) \in K \\ \text{and} \\ \|F_j\|_{L_t^q L_x^p(\mathbb{R}_+^{1+n})} < 2^{-j} \end{cases}$$

derives that  $F = \sum_{j=1}^{\infty} F_j$  enjoys the required properties.

(ii) This follows from the definition of  $(\alpha, p, q)$ -capacity.

(iii) The forthcoming argument is standard; see also [1, 16, 3].

*Case 1:  $p \geq q$ .* If we choose  $F_j$  with  $S_\alpha F_j \geq 1$  on  $K_j$ , then  $F = \sup_{j=1,2,3,\dots} F_j$  satisfies  $S_\alpha F \geq 1$  on  $\bigcup_{j=1}^{\infty} K_j$  and

$$\|F\|_{L_t^q L_x^p(\mathbb{R}_+^{1+n})}^q \leq \int_0^\infty \left( \sum_{j=1}^\infty \int_{\mathbb{R}^n} |F_j|^p dx \right)^{\frac{q}{p}} dt \leq \sum_{j=1}^\infty \int_0^\infty \left( \int_{\mathbb{R}^n} |F_j|^p dx \right)^{\frac{q}{p}} dt.$$

So, the desired inequality follows.

*Case 2:*  $p < q$ . Now, the Minkowski inequality implies that

$$\begin{aligned} \|F\|_{L_t^q L_x^p(\mathbb{R}_+^{1+n})}^p &\leq \left[ \int_0^\infty \left( \sum_{j=1}^\infty \int_{\mathbb{R}^n} |F_j|^p dx \right)^{\frac{q}{p}} dt \right]^{\frac{p}{q}} \\ &\leq \sum_{j=1}^\infty \left[ \int_0^\infty \left( \int_{\mathbb{R}^n} |F_j|^p dx \right)^{\frac{q}{p}} dt \right]^{\frac{p}{q}}, \end{aligned}$$

whence deducing the desired inequality.

(iv) This is a consequence of the following implication:

$$F_{x_0}(t, x) = F(t, x + x_0) \implies \|F_{x_0}\|_{L_t^q L_x^p(\mathbb{R}_+^{1+n})} = \|F\|_{L_t^q L_x^p(\mathbb{R}_+^{1+n})},$$

which completes the proof of Proposition 2.2.  $\square$

### 3. PROOFS OF THEOREMS 1.2-1.3 AND COROLLARY 1.4

Now, we are ready to carry out the task as just mentioned in the title of Section 3.

**3.1. Proof of Theorem 1.2.** (i) Let

$$\begin{cases} (t, x) \in \mathbb{R}_+^{1+n}; \\ (t_0, x_0) \in \mathbb{R}_+^{1+n}; \\ f \in L^p(\mathbb{R}^n); \\ p \in [1, \infty]; \\ 0 \leq t_1 < t_2 < \infty. \end{cases}$$

Since  $K_{t_0}^{(\alpha)}(\cdot)$  is of  $C^\infty(\mathbb{R}^n)$ , one has that  $R_\alpha f(t_0, x) = e^{-t_0(-\Delta)^\alpha} f(x)$  is of  $C^\infty(\mathbb{R}^n)$  too. Meanwhile, for  $x \in \mathbb{R}^n$  one gets

$$R_\alpha f(t_1, x) - R_\alpha f(t_2, x) = \int_{t_1}^{t_2} (-\Delta)^\alpha e^{-t(-\Delta)^\alpha} f(x) dt.$$

Note that the kernel  $\tilde{K}_t^{(\alpha)}(\cdot)$  of  $(-\Delta)^\alpha e^{-t(-\Delta)^\alpha}$  obeys  $|\tilde{K}_t^{(\alpha)}(x)| \lesssim (t^{\frac{1}{2\alpha}} + |x|)^{-n-2\alpha}$ ; see also [8, Lemma 2.2 & (2.5)]. So, an application of [8, Lemma 3.1] gives

$$\left\| (-\Delta)^\alpha e^{-t(-\Delta)^\alpha} f \right\|_{L^\infty(\mathbb{R}^n)} \lesssim \begin{cases} t^{-1-\frac{n}{2\alpha p}} \|f\|_{L^p(\mathbb{R}^n)} & \text{for } p \in [1, \infty); \\ t^{-1} \|f\|_{L^\infty(\mathbb{R}^n)} & \text{for } p = \infty, \end{cases}$$

and hence

$$|R_\alpha f(t_1, x) - R_\alpha f(t_2, x)| \lesssim \|f\|_{L^p(\mathbb{R}^n)} \begin{cases} |t_1^{-1-\frac{n}{2\alpha p}} - t_2^{-1-\frac{n}{2\alpha p}}| & \text{for } p \in [1, \infty); \\ |\ln t_1 - \ln t_2| & \text{for } p = \infty. \end{cases}$$

Putting the above facts together yields

$$|R_\alpha f(t, x) - R_\alpha f(t_0, x_0)| \leq |R_\alpha f(t_0, x) - R_\alpha f(t_0, x_0)| + |R_\alpha f(t, x) - R_\alpha f(t_0, x)|$$

$$\rightarrow 0 \text{ as } (t, x) \rightarrow (t_0, x_0).$$

Therefore  $R_\alpha f$  is of  $C(\mathbb{R}_+^{1+n})$ .

(ii) Let  $(t, x) \in \mathbb{R}_+^{1+n}$  be fixed. Then we have

$$|S_\alpha F(t, x)| \leq \int_0^t \int_{\mathbb{R}^n} K_{t-s}^{(\alpha)}(x-y) |F(s, y)| dy ds = \text{I} + \text{II},$$

where

$$\begin{cases} \text{I} := \int_0^r \int_{\mathbb{R}^n} K_{t-s}^{(\alpha)}(x-y) |F(s, y)| dy ds; \\ \text{II} := \int_r^t \int_{\mathbb{R}^n} K_{t-s}^{(\alpha)}(x-y) |F(s, y)| dy ds. \end{cases}$$

From the Hölder inequality and the assumption  $\frac{n}{p} + \frac{2\alpha}{q} = 2\alpha$  it follows that

$$\begin{aligned} \text{I} &\lesssim \int_0^r \int_{\mathbb{R}^n} \frac{|t-s|}{(|t-s|^{\frac{1}{2\alpha}} + |x-y|)^{n+2\alpha}} |F(s, y)| dy ds \\ &\lesssim \int_0^r |t-s| \|F(s, \cdot)\|_{L_x^p(\mathbb{R}^n)} \left( \int_{\mathbb{R}^n} \frac{dy}{(|t-s|^{\frac{1}{2\alpha}} + |x-y|)^{(n+2\alpha)(\frac{p}{p-1})}} \right)^{\frac{p-1}{p}} ds \\ &\lesssim \int_0^r \frac{\|F(s, \cdot)\|_{L_x^p(\mathbb{R}^n)}}{|t-s|^{\frac{n}{2p\alpha}}} ds \\ &\lesssim \|F\|_{L_t^q L_x^p(\mathbb{R}_+^{1+n})} \left( \int_0^r \frac{ds}{|t-s|^{\frac{n}{2p\alpha}(\frac{q}{q-1})}} \right)^{\frac{q-1}{q}} \\ &\lesssim \|F\|_{L_t^q L_x^p(\mathbb{R}_+^{1+n})} \left( \ln \frac{t}{t-r} \right)^{\frac{q-1}{q}}. \end{aligned}$$

Similarly, by using  $M_{\mathbb{R}}$  - the Hardy-Littlewood maximal function on  $\mathbb{R}$ , we obtain

$$\begin{aligned} \text{II} &\lesssim \int_r^t \int_{\mathbb{R}^n} \frac{|t-s|}{(|t-s|^{\frac{1}{2\alpha}} + |x-y|)^{n+2\alpha}} |F(s, y)| dy ds \\ &\lesssim \int_r^t \frac{\|F(s, \cdot)\|_{L_x^p(\mathbb{R}^n)}}{|t-s|^{\frac{n}{2p\alpha}}} ds \\ &\lesssim \sum_{k=0}^{-\infty} \int_{t-2^k|t-r|}^{t-2^{k-1}|t-r|} \frac{\|F(s, \cdot)\|_{L_x^p(\mathbb{R}^n)}}{|t-s|^{\frac{n}{2p\alpha}}} ds \\ &\lesssim \sum_{k=0}^{-\infty} \frac{1}{(2^k|t-r|)^{\frac{n}{2p\alpha}}} \int_{t-2^k|t-r|}^t \|F(s, \cdot)\|_{L_x^p(\mathbb{R}^n)} ds \\ &\lesssim \sum_{k=0}^{-\infty} (2^k|t-r|)^{1-\frac{n}{2p\alpha}} M_{\mathbb{R}}(\|F(\cdot, \cdot)\|_{L_x^p(\mathbb{R}^n)})(t) \\ &\lesssim |t-r|^{1/q} M_{\mathbb{R}}(\|F(\cdot, \cdot)\|_{L_x^p(\mathbb{R}^n)})(t). \end{aligned}$$

Via choosing  $r \in (0, t)$  such that

$$|t - r|^{1/q} = \min \left\{ t^{1/q}, \frac{\|F\|_{L_t^q L_x^p(\mathbb{R}_+^{1+n})}}{\mathbf{M}_{\mathbb{R}}(\|F(\cdot, \cdot)\|_{L_x^p(\mathbb{R}^n)})(t)} \right\},$$

we see that

$$|S_{\alpha} F(t, x)| \lesssim \|F\|_{L_t^q L_x^p(\mathbb{R}_+^{1+n})} \max \left\{ 1, \left[ \ln \frac{et^{1/q} \mathbf{M}_{\mathbb{R}}(\|F\|_{L_x^p(\mathbb{R}^n)})(t)}{\|F\|_{L_t^q L_x^p(\mathbb{R}_+^{1+n})}} \right]^{\frac{q-1}{q}} \right\}.$$

Letting  $r_0 = t_0^{\frac{1}{2\alpha}}$  yields a constant  $C > 0$  such that

$$\begin{aligned} & \iint_{B_{r_0}^{(\alpha)}(t_0, x_0)} \exp \left( \frac{S_{\alpha} F(t, x)}{C \|F\|_{L_t^q L_x^p(\mathbb{R}_+^{1+n})}} \right)^{\frac{q}{q-1}} dx dt \\ & \lesssim \iint_{B_{r_0}^{(\alpha)}(t_0, x_0)} \frac{et^{1/q} \mathbf{M}_{\mathbb{R}}(\|F\|_{L_x^p(\mathbb{R}^n)})(t)}{\|F\|_{L_t^q L_x^p(\mathbb{R}_+^{1+n})}} dx dt \\ & \lesssim r_0^n t_0^{1/q} \int_0^{2t_0} \frac{\mathbf{M}_{\mathbb{R}}(\|F\|_{L_x^p(\mathbb{R}^n)})(t)}{\|F\|_{L_t^q L_x^p(\mathbb{R}_+^{1+n})}} dt \\ & \lesssim t_0^{1/q} r_0^{n+2\alpha-2\alpha/q} \\ & \approx |B_{r_0}^{(\alpha)}(t_0, x_0)|, \end{aligned}$$

which completes the proof of (ii).

(iii) Given a point  $(t_0, x_0) \in \mathbb{R}_+^{1+n}$ , let  $x \in \mathbb{R}^n$  be sufficient close to  $x_0$  and  $\delta = |x - x_0|$ . Then

$$\begin{aligned} & |S_{\alpha} F(t_0, x_0) - S_{\alpha} F(t_0, x)| \\ & \leq \int_0^{t_0} \int_{\mathbb{R}^n} |K_{t_0-s}^{(\alpha)}(x_0 - y) - K_{t_0-s}^{(\alpha)}(x - y)| |F(y, s)| dy ds \\ & \leq \int_0^{t_0} \int_{B(x_0, 3\delta)} \cdots dy ds + \int_0^{t_0} \int_{\mathbb{R}^n \setminus B(x_0, 3\delta)} \cdots dy ds \\ & =: \text{I} + \text{II}. \end{aligned}$$

Note that

$$\begin{aligned} & \int_0^{t_0} \int_{B(x_0, 3\delta)} K_{t_0-s}^{(\alpha)}(x_0 - y) |F(y, s)| dy ds \\ & \leq \int_0^{t_0 - (2\delta)^{2\alpha}} \int_{B(x_0, 3\delta)} \left( \frac{|t - s|}{|t - s|^{1 + \frac{n}{2\alpha}}} \right) |F(y, s)| dy ds \\ & \quad + \int_{t_0 - (2\delta)^{2\alpha}}^{t_0} \int_{B(x_0, 3\delta)} \left( \frac{|t - s|}{(|t - s|^{\frac{1}{2\alpha}} + |x_0 - y|)^{n+\alpha}} \right) |F(y, s)| dy ds \end{aligned}$$

$$\begin{aligned}
&\lesssim \int_0^{t_0-(2\delta)^{2\alpha}} \left( \frac{\delta^{\frac{n(p-1)}{p}}}{|t-s|^{\frac{n}{2\alpha}}} \right) \|F(\cdot, s)\|_{L_x^p(\mathbb{R}^n)} ds \\
&\quad + \int_{t_0-(2\delta)^{2\alpha}}^t \frac{\|F(\cdot, s)\|_{L_x^p(\mathbb{R}^n)}}{|t-s|^{\frac{n}{2p\alpha}}} ds \\
&\lesssim \|F\|_{L_t^q L_x^p(\mathbb{R}_+^{1+n})} \delta^{\frac{n(p-1)}{p}} \left( \int_0^{t_0-(2\delta)^{2\alpha}} \frac{ds}{|t-s|^{\frac{nq}{2\alpha(q-1)}}} \right)^{\frac{q-1}{q}} \\
&\quad + \|F\|_{L_t^q L_x^p(\mathbb{R}_+^{1+n})} \left( \int_{t_0-(2\delta)^{2\alpha}}^t \frac{ds}{|t-s|^{\frac{nq}{2p\alpha(q-1)}}} \right)^{\frac{q-1}{q}} \\
&\lesssim \|F\|_{L_t^q L_x^p(\mathbb{R}_+^{1+n})} \left( \delta^{\frac{n(p-1)}{p}} \delta^{\frac{2\alpha(q-1)}{q}-n} + \delta^{\frac{2\alpha(q-1)}{q}-\frac{n}{p}} \right) \\
&\lesssim \|F\|_{L_t^q L_x^p(\mathbb{R}_+^{1+n})} \delta^{\frac{2\alpha(q-1)}{q}-\frac{n}{p}}.
\end{aligned}$$

Thus the first term I is bounded from above as

$$\begin{aligned}
\text{I} &\leq \int_0^{t_0} \int_{B(x_0, 3\delta)} K_{t_0-s}^{(\alpha)}(x_0 - y) \|F(y, s)\| dy ds \\
&\quad + \int_0^{t_0} \int_{B(x, 4\delta)} K_{t_0-s}^{(\alpha)}(x - y) \|F(y, s)\| dy ds \\
&\lesssim \|F\|_{L_t^q L_x^p(\mathbb{R}_+^{1+n})} |x - x_0|^{\frac{2\alpha(q-1)}{q}-\frac{n}{p}}.
\end{aligned}$$

To estimate the second term II, notice that

$$|\nabla K_1^{(\alpha)}(x)| \lesssim (1 + |x|)^{-n-1};$$

see also [8, Remark 2.1]. Using this and the Hölder inequality, we have

$$\begin{aligned}
\text{II} &\leq \int_0^{t_0} \int_{\mathbb{R}^n \setminus B(x_0, 3\delta)} |K_{t_0-s}^{(\alpha)}(x_0 - y) - K_{t_0-s}^{(\alpha)}(x - y)| \|F(y, s)\| dy ds \\
&\leq \int_0^{t_0} \int_{\mathbb{R}^n \setminus B(x_0, 3\delta)} \left( \frac{\delta}{|t-s|^{\frac{1}{2\alpha}}} \right) \left( \frac{|t-s|^{\frac{1}{2\alpha}}}{(|t-s|^{\frac{1}{2\alpha}} + |x_0 - y|)^{n+1}} \right) \|F(y, s)\| dy ds \\
&\lesssim \int_0^{t_0-(2\delta)^{2\alpha}} \int_{\mathbb{R}^n \setminus B(x_0, 3\delta)} \left( \frac{\delta}{(|t-s|^{\frac{1}{2\alpha}} + |x_0 - y|)^{n+1}} \right) \|F(y, s)\| dy ds \\
&\quad + \int_{t_0-(2\delta)^{2\alpha}}^t \int_{\mathbb{R}^n \setminus B(x_0, 3\delta)} \left( \frac{\delta}{|x_0 - y|^{n+1}} \right) \|F(y, s)\| dy ds \\
&\lesssim \int_0^{t_0-(2\delta)^{2\alpha}} \delta \|F(\cdot, s)\|_{L_x^p(\mathbb{R}^n)} |t-s|^{-\frac{n}{2\alpha p}-\frac{1}{2\alpha}} ds \\
&\quad + \int_{t_0-(2\delta)^{2\alpha}}^t \|F(\cdot, s)\|_{L_x^p(\mathbb{R}^n)} \delta^{-\frac{n}{p}} ds
\end{aligned}$$

$$\lesssim \|F\|_{L_t^q L_x^p(\mathbb{R}_+^{1+n})} |x - x_0|^{\frac{2\alpha(q-1)}{q} - \frac{n}{p}}.$$

Thus, we conclude that

$$|S_\alpha F(t_0, x_0) - S_\alpha F(t_0, x)| \lesssim \|F\|_{L_t^q L_x^p(\mathbb{R}_+^{1+n})} |x - x_0|^{\frac{2\alpha(q-1)}{q} - \frac{n}{p}}.$$

Let  $(x, t_1), (x, t_2) \in \mathbb{R}_+^{1+n}$ . Without loss of generality we may assume  $t_1 > t_2$ , and then write

$$\begin{aligned} |S_\alpha F(t_1, x) - S_\alpha F(t_2, x)| &\leq \int_0^{t_2} |(e^{-(t_1-s)(-\Delta)^\alpha} - e^{-(t_2-s)(-\Delta)^\alpha})F(x, s)| ds \\ &\quad + \int_{t_2}^{t_1} |(e^{-(t_1-s)(-\Delta)^\alpha})F(x, s)| ds \\ &=: \text{III} + \text{IV}. \end{aligned}$$

By using the mapping property of the semigroup, we obtain

$$\begin{aligned} \text{III} &\leq \int_0^{t_2} \int_{t_2-s}^{t_1-s} |(-\Delta)^\alpha e^{-r(-\Delta)^\alpha} F(x, s)| dr ds \\ &\leq \int_0^{t_2} \int_{t_2-s}^{t_1-s} r^{-1-\frac{n}{2\alpha p}} \|F(\cdot, s)\|_{L_x^p(\mathbb{R}^n)} dr ds \\ &\leq \int_0^{t_2} \int_0^{t_1-t_2} (t_2 - s + r)^{-1-\frac{n}{2\alpha p}} \|F(\cdot, s)\|_{L_x^p(\mathbb{R}^n)} ds \\ &\leq \int_0^{t_1-t_2} \int_0^{t_2} (t_2 - s + r)^{-1-\frac{n}{2\alpha p}} \|F(\cdot, s)\|_{L_x^p(\mathbb{R}^n)} ds \\ &\lesssim \|F\|_{L_t^q L_x^p(\mathbb{R}_+^{1+n})} \int_0^{t_1-t_2} r^{\frac{q-1}{q}-1-\frac{n}{2\alpha p}} dr \\ &\lesssim |t_2 - t_1|^{1-\frac{1}{q}-\frac{n}{2\alpha p}} \|F\|_{L_t^q L_x^p(\mathbb{R}_+^{1+n})}, \end{aligned}$$

and

$$\begin{aligned} \text{IV} &\leq \int_{t_2}^{t_1} (t_1 - s)^{-\frac{n}{2\alpha p}} \|F(s, \cdot)\|_{L_x^p(\mathbb{R}^n)} ds \\ &\lesssim |t_2 - t_1|^{1-\frac{1}{q}-\frac{n}{2\alpha p}} \|F\|_{L_t^q L_x^p(\mathbb{R}_+^{1+n})}. \end{aligned}$$

Hence

$$|S_\alpha F(t_1, x) - S_\alpha F(t_2, x)| \lesssim |t_2 - t_1|^{1-\frac{1}{q}-\frac{n}{2\alpha p}} \|F\|_{L_t^q L_x^p(\mathbb{R}_+^{1+n})}$$

The difference estimates on  $S_\alpha$  give us that if  $(t, x)$  is close to  $(t_0, x_0)$  then

$$\begin{aligned} &|S_\alpha F(t, x) - S_\alpha F(t_0, x_0)| \\ &\leq |S_\alpha F(t, x) - S_\alpha F(t_0, x)| + |S_\alpha F(t_0, x) - S_\alpha F(t_0, x_0)| \\ &\lesssim \left( |t - t_0|^{1-\frac{1}{q}-\frac{n}{2\alpha p}} + |x - x_0|^{\frac{2\alpha(q-1)}{q} - \frac{n}{p}} \right) \|F\|_{L_t^q L_x^p(\mathbb{R}_+^{1+n})}, \end{aligned}$$

which completes the proof of (iii).

**3.2. Proof of Theorem 1.3.** (i) In the sequel, let  $\beta = (p \wedge q)(\frac{n}{p} + \frac{2\alpha}{q} - 2\alpha)$ . Also, assume that  $F \geq 0$  satisfies  $S_\alpha F \geq 1_{B_{r_0}^{(\alpha)}(0,0)}$ . Then, according to the definition of operator  $S_\alpha$ , the following transform

$$\begin{cases} s = \frac{t}{r_0^{2\alpha}}; \\ y = \frac{x}{r_0}; \\ F_{r_0}(s, y) = F(r_0^{2\alpha}s, r_0y); \\ G(s, y) = r_0^{2\alpha}F_{r_0}(s, y), \end{cases}$$

enjoys the property  $S_\alpha G \geq 1_{B_1^{(\alpha)}(0,0)}$ . Thus,

$$C_{p,q}^{(\alpha)}(B_1^{(\alpha)}(0,0)) \leq \|r_0^{2\alpha}F_{r_0}\|_{L_t^q L_x^p(\mathbb{R}_+^{1+n})}^{p \wedge q} = r_0^{-\beta} \|F\|_{L_t^q L_x^p(\mathbb{R}_+^{1+n})}^{p \wedge q}.$$

This implies that

$$C_{p,q}^{(\alpha)}(B_1^{(\alpha)}(0,0)) \leq r_0^{-\beta} C_{p,q}^{(\alpha)}(B_{r_0}^{(\alpha)}(0,0)).$$

In fact, the last inequality is an equality since changing the order of  $B_1^{(\alpha)}(0,0)$  and  $B_{r_0}^{(\alpha)}(0,0)$  derives

$$C_{p,q}^{(\alpha)}(B_{r_0}^{(\alpha)}(0,0)) \leq r_0^\beta C_{p,q}^{(\alpha)}(B_1^{(\alpha)}(0,0)).$$

Next, we consider the desired equivalence estimate. If  $F \geq 0$  and  $S_\alpha F \geq 1_{B_{r_0}^{(\alpha)}(t_0, x_0)}$ , then, for  $1 \leq p < \infty$  and  $1 < q < \infty$ , there exist  $\tilde{p}$  and  $\tilde{q}$  such that

$$\begin{cases} 1 \leq p < \tilde{p} < \infty; \\ 1 < q < \tilde{q} < \infty; \\ \left(\frac{1}{\tilde{q}} - \frac{1}{q}\right) + \frac{n}{2\alpha} \left(\frac{1}{\tilde{p}} - \frac{1}{p}\right) = 1. \end{cases}$$

Consequently, according to Theorem 1.1(ii) we have

$$\|S_\alpha F\|_{L_t^{\tilde{q}} L_x^{\tilde{p}}(\mathbb{R}_+^{1+n})} \lesssim \|F\|_{L_t^q L_x^p(\mathbb{R}_+^{1+n})}.$$

This, along with the definition of  $C_{p,q}^{(\alpha)}(\cdot)$ , implies that

$$r_0^\beta \lesssim C_{p,q}^{(\alpha)}(B_{r_0}^{(\alpha)}(t_0, x_0))$$

thanks to

$$\frac{n}{\tilde{p}} + \frac{2\alpha}{\tilde{q}} = \frac{n}{p} + \frac{2\alpha}{q} - 2\alpha.$$

To get the corresponding upper bound of  $C_{p,q}^{(\alpha)}(B_{r_0}^{(\alpha)}(t_0, x_0))$ , we consider

$$B_{r_0, \eta}^{(\alpha)}(t_0, x_0) := \{(t, x) \in \mathbb{R}_+^{1+n} : |t - t_0| < (\eta r_0)^{2\alpha} \text{ \& \> } |x - x_0| < r_0\}$$

for some sufficiently large  $\eta > 0$  which will be determined later. Note that  $(t, x) \in B_{r_0}^{(\alpha)}(t_0, x_0)$  ensures

$$\begin{aligned}
& S_\alpha 1_{B_{r_0, \eta}^{(\alpha)}(t_0, x_0)}(t, x) \\
&= \int_0^t \int_{\mathbb{R}^n} K_{t-s}^{(\alpha)}(x-y) 1_{B_{r_0, \eta}^{(\alpha)}(t_0, x_0)}(s, y) dy ds \\
&= \int_{(0, t) \cap \{s: |s-t_0| < (\eta r_0)^{2\alpha}\}} \int_{|y-x_0| < r_0} K_{t-s}^{(\alpha)}(x-y) dy ds \\
&\geq \int_{(0, t) \cap \{s: |s-t_0| < (\eta r_0)^{2\alpha}\} \cap \{s: t-s > \frac{\eta^{2\alpha}-1}{2} r_0^{2\alpha}\}} \int_{|y-x_0| < r_0} K_{t-s}^{(\alpha)}(x-y) dy ds
\end{aligned}$$

for sufficiently small  $r_0 > 0$ . According to [13, Proposition 1], there are positive constants  $\sigma$  and  $\kappa$ , depending only  $n$  and  $\alpha$ , such that

$$\inf\{K_t^{(\alpha)}(x) : |x| \leq \sigma t^{\frac{1}{2\alpha}}\} \geq \kappa t^{-\frac{n}{2\alpha}}.$$

Under

$$\begin{cases} (t, x) \in B_{r_0}^{(\alpha)}(t_0, x_0); \\ |y - x_0| < r_0; \\ t - s > \left(\frac{\eta^{2\alpha}-1}{2}\right) r_0^{2\alpha}, \end{cases}$$

one has

$$|x - y| \leq |x - x_0| + |y - x_0| < 2r_0 < 2 \left( \frac{2}{\eta^{2\alpha} - 1} \right)^{\frac{1}{2\alpha}} |t - s|^{\frac{1}{2\alpha}} < \sigma |t - s|^{\frac{1}{2\alpha}}$$

for some large enough  $\eta$  with

$$2 \left( \frac{2}{\eta^{2\alpha} - 1} \right)^{\frac{1}{2\alpha}} < \sigma.$$

Thus, one gets that if  $|t - t_0| < r^{2\alpha}$  then

$$\begin{aligned}
& S_\alpha 1_{B_{r_0, \eta}^{(\alpha)}(t_0, x_0)}(t, x) \\
&\geq \int_{(0, t) \cap \{s: |s-t_0| < (\eta r_0)^{2\alpha}\} \cap \{s: t-s > (\eta^{2\alpha}-1)2^{-1} r_0^{2\alpha}\}} \int_{|y-x_0| < r_0} |t-s|^{-\frac{n}{2\alpha}} dy ds \\
&\geq c r_0^{2\alpha}
\end{aligned}$$

holds for some constant  $c > 0$  independent of  $r_0$ . Consequently,

$$S_\alpha \left( \frac{1_{B_{r_0, \eta}^{(\alpha)}(t_0, x_0)}}{c r_0^{2\alpha}} \right)(t, x) \geq 1, \quad \forall (t, x) \in B_{r_0}^{(\alpha)}(t_0, x_0).$$

This gives

$$C_{p,q}^{(\alpha)}(B_{r_0}^{(\alpha)}(t_0, x_0)) \leq \left\| \frac{1_{B_{r_0, \eta}^{(\alpha)}(t_0, x_0)}}{c r_0^{2\alpha}} \right\|_{L_t^q L_x^p(\mathbb{R}_+^{1+n})}^{p \wedge q} \lesssim r_0^\beta.$$



(ii) For an arbitrarily fixed point  $(t_0, x_0) \in \mathbb{R}_+^{1+n}$ . Let  $r_0 \ll \min\{t_0, 1\}$ . Suppose that  $S_\alpha F(t, x) \geq 1$  on  $B_{r_0}^{(\alpha)}(t_0, x_0)$ . Then by Theorem 1.2(ii), we have a constant  $C > 0$  such that

$$\begin{aligned} & \iint_{B_{r_0}^{(\alpha)}(t_0, x_0)} \exp\left(\frac{S_\alpha F(t, x)}{C\|F\|_{L_t^q L_x^p(\mathbb{R}_+^{1+n})}}\right)^{\frac{q}{q-1}} dx dt \\ & \lesssim \iint_{B_{r_0}^{(\alpha)}(t_0, x_0)} \frac{et^{1/q} \mathbf{M}_{\mathbb{R}}(\|F\|_{L_x^p(\mathbb{R}^n)})(t)}{\|F\|_{L_t^q L_x^p(\mathbb{R}_+^{1+n})}} dx dt \\ & \lesssim r_0^{n+2\alpha} + r_0^n t_0^{1/q} \int_{t_0-r_0^{2\alpha}}^{t_0+r_0^{2\alpha}} \frac{\mathbf{M}_{\mathbb{R}}(\|F\|_{L_x^p(\mathbb{R}^n)})(t)}{\|F\|_{L_t^q L_x^p(\mathbb{R}_+^{1+n})}} dt \\ & \lesssim t_0^{1/q} r_0^{n+2\alpha-2\alpha/q}. \end{aligned}$$

On the other hand, as  $S_\alpha F(\cdot, \cdot) \geq 1$  on  $B_{r_0}^{(\alpha)}(t_0, x_0)$ , it follows that for a constant  $c > 0$ ,

$$\iint_{B_{r_0}^{(\alpha)}(t_0, x_0)} \exp\left(\frac{S_\alpha F(t, x)}{c[\ln \frac{1}{r_0}]^{\frac{1-q}{q}}}\right)^{\frac{q}{q-1}} dx dt \gtrsim r_0^{n+2\alpha} \exp\left(c^{-\frac{q}{q-1}} \ln \frac{1}{r_0}\right) \gtrsim r_0^{n+2\alpha-c^{-\frac{q}{q-1}}},$$

which implies that

$$\|F\|_{L_t^q L_x^p(\mathbb{R}_+^{1+n})} \gtrsim \left(\ln \frac{1}{r_0}\right)^{\frac{1-q}{q}}$$

and hence

$$C_{p,q}^{(\alpha)}(B_{r_0}^{(\alpha)}(t_0, x_0)) \gtrsim \left(\ln \frac{1}{r_0}\right)^{\frac{1-q}{q}(p \wedge q)} \quad \text{as } r_0 \rightarrow 0.$$

Next, we prove the converse form of the last inequality. Let

$$E := \{(t, x) \in \mathbb{R}_+^{1+n} : (2r_0)^{2\alpha} < t_0 - t < (2r_0)^\alpha \text{ \& } |t - t_0|^{\frac{1}{2\alpha}} < |x_0 - x| < 2\}.$$

Define

$$F(x, t) := \begin{cases} \frac{1}{(|t_0 - t|^{\frac{1}{2\alpha}} + |x - x_0|)^{2\alpha}}, & \forall (t, x) \in E; \\ 0, & \text{otherwise.} \end{cases}$$

By using the known estimate below

$$K_t^{(\alpha)}(x - y) \approx \frac{t}{(t^{\frac{1}{2\alpha}} + |x - y|)^{n+2\alpha}},$$

we see that for each  $(t, x) \in B_{r_0}^{(\alpha)}(t_0, x_0)$ ,

$$\begin{aligned} S_\alpha F(t, x) &= \int_0^t \int_{\mathbb{R}^n} K_{t-s}^{(\alpha)}(x - y) F(y, s) dy ds \\ &\approx \iint_E \frac{|t - s|}{(|t - s|^{\frac{1}{2\alpha}} + |x - y|)^{n+2\alpha}} F(y, s) dy ds \end{aligned}$$

$$\begin{aligned}
&\approx \iint_E \frac{|t_0 - s|}{(|t_0 - s|^{\frac{1}{2\alpha}} + |x_0 - y|)^{n+2\alpha}} F(y, s) dy ds \\
&\gtrsim \int_{t_0 - (2r_0)^\alpha}^{t_0 - (2r_0)^{2\alpha}} \int_{B(x_0, 2) \setminus B(x_0, |t_0 - s|^{\frac{1}{2\alpha}})} \frac{|t_0 - s|}{(|t_0 - s|^{\frac{1}{2\alpha}} + |x_0 - y|)^{n+4\alpha}} dy ds \\
&\gtrsim \int_{t_0 - (2r_0)^\alpha}^{t_0 - (2r_0)^{2\alpha}} \int_{|t_0 - s|^{\frac{1}{2\alpha}}}^2 \frac{|t_0 - s| r^{n-1}}{(|t_0 - s|^{\frac{1}{2\alpha}} + r)^{n+4\alpha}} dr ds \\
&\gtrsim \int_{t_0 - (2r_0)^\alpha}^{t_0 - (2r_0)^{2\alpha}} |t_0 - s| \left( |t_0 - s|^{-2} - 2^{-4\alpha} \right) ds \\
&\gtrsim \int_{t_0 - (2r_0)^\alpha}^{t_0 - (2r_0)^{2\alpha}} |t_0 - s|^{-1} ds \\
&\gtrsim \ln \frac{1}{(2r_0)^\alpha}.
\end{aligned}$$

Moreover, noticing that  $2p\alpha > n$ , we have

$$\begin{aligned}
\|F\|_{L_t^q L_x^p(\mathbb{R}_+^{1+n})}^q &\lesssim \int_{t_0 - (2r_0)^\alpha}^{t_0 - (2r_0)^{2\alpha}} \left( \int_{B(x_0, 2) \setminus B(x_0, |t_0 - s|^{\frac{1}{2\alpha}})} \frac{1}{(|t_0 - s|^{\frac{1}{2\alpha}} + |x_0 - y|)^{2p\alpha}} dy \right)^{q/p} ds \\
&\lesssim \int_{t_0 - (2r_0)^\alpha}^{t_0 - (2r_0)^{2\alpha}} \left( \int_{|t_0 - s|^{\frac{1}{2\alpha}}}^2 r^{n-1-2p\alpha} dr \right)^{q/p} ds \\
&\lesssim \int_{t_0 - (2r_0)^\alpha}^{t_0 - (2r_0)^{2\alpha}} |t_0 - s|^{\frac{(n-2p\alpha)q}{2p\alpha}} ds \\
&\lesssim \ln \frac{1}{(2r_0)^\alpha}.
\end{aligned}$$

The above two estimates give

$$C_{p,q}^{(\alpha)}(B_{r_0}^{(\alpha)}(t_0, x_0)) \lesssim \left\| \frac{F}{\ln \frac{1}{r_0}} \right\|_{L_t^q L_x^p(\mathbb{R}_+^{1+n})}^{q \wedge p} \lesssim \left( \ln \frac{1}{r_0} \right)^{\frac{1-q}{q}(p \wedge q)} \quad \text{as } r_0 \rightarrow 0.$$

**3.3. Proof of Corollary 1.4.** (i) This follows from Theorem 1.1(ii), Theorem 1.3(i) and Proposition 2.2(iii).

(ii)-(iii) The comparison inequalities for the two capacities follow from Proposition 2.2(iii) and (i)-(ii) of Theorem 1.3. To get the dimension inequality, we firstly keep in mind the fact

$$C_{p,q}^{(\alpha)}(\mathcal{B}[S_\alpha F; p, q]) = 0 \quad \text{for } 0 \leq F \in L_t^q L_x^p(\mathbb{R}_+^{1+n}),$$

and secondly recall Proposition 2.1 and the following Frostman type theorem (cf. [2, Theorem 5.1.12]): if  $\phi : [0, \infty) \mapsto [0, \infty]$  increases with

$\phi(0) = 0$  then for a given compact  $K \subset \mathbb{R}_+^{1+n}$  there is a measure  $\mu \in \mathcal{M}^+(K)$  obeying  $\mu(B_r^{(\alpha)}(t, x)) \lesssim \phi(r)$  such that  $\mu(K) \approx H_\infty^{\phi, \alpha}(K)$ .

Now, let  $K$  be any compact subset of the blow-up set  $\mathcal{B}[S_\alpha F; p, q]$  and be contained in a ball  $B_R^{(\alpha)}(t_0, x_0)$ . Taking  $0 \leq G \in L_t^q L_x^p(\mathbb{R}_+^{1+n})$  such that  $S_\alpha G \geq 1$  on  $K$ , we use the dyadic decomposition of a set and the Hölder inequality to get that if  $0 < R_0 < 1 \wedge R$  then

$$\begin{aligned}
\mu(K) &\leq \iint_K S_\alpha G(t, x) d\mu(t, x) \\
&\leq \iint_{\mathbb{R}_+^{1+n}} G(s, y) \iint_{K \cap ((s, \infty) \times \mathbb{R}^n)} K_{t-s}^{(\alpha)}(x - y) d\mu(t, x) dy ds \\
&\lesssim \iint_{\mathbb{R}_+^{1+n}} G(s, y) \iint_{K \cap ((s, \infty) \times \mathbb{R}^n)} \left( \frac{|t - s|}{(|t - s|^{\frac{1}{2\alpha}} + |x - y|)^{n+2\alpha}} \right) d\mu(t, x) dy ds \\
&\lesssim \mu(K) \iint_{\mathbb{R}_+^{1+n} \setminus B_{2R_0}^{(\alpha)}(t_0, x_0)} G(s, y) \left( \frac{|t_0 - s|}{(|t_0 - s|^{\frac{1}{2\alpha}} + |x_0 - y|)^{n+2\alpha}} \right) dy ds \\
&\quad + \iint_{B_{2R_0}^{(\alpha)}(t_0, x_0)} G(s, y) \left( \sum_{j=0}^{\infty} \frac{\mu(B_{2^{-j}R_0}^{(\alpha)}(s, y))}{(2^{-j}R_0)^n} \right) dy ds \\
&\lesssim \mu(K) R_0^{2\alpha - \frac{2\alpha}{q} - \frac{n}{p}} \|G\|_{L_t^q L_x^p(\mathbb{R}_+^{1+n})} \\
&\quad + \iint_{B_{2R_0}^{(\alpha)}(t_0, x_0)} G(s, y) \int_0^{R_0} \frac{\mu(B_r^{(\alpha)}(s, y))}{r^{1+n}} dr dy ds \\
&\lesssim \mu(K) R_0^{2\alpha - \frac{2\alpha}{q} - \frac{n}{p}} \|G\|_{L_t^q L_x^p(\mathbb{R}_+^{1+n})} \\
&\quad + \int_0^{R_0} \iint_{B_{2R_0}^{(\alpha)}(t_0, x_0)} G(s, y) \mu(B_r^{(\alpha)}(s, y)) dy ds \frac{dr}{r^{1+n}}.
\end{aligned}$$

For  $p \leq q$ , we have

$$\begin{aligned}
&\int_0^{R_0} \iint_{B_{2R_0}^{(\alpha)}(t_0, x_0)} G(s, y) \mu(B_r^{(\alpha)}(s, y)) dy ds \frac{dr}{r^{1+n}} \\
&\leq \|G\|_{L_t^p L_x^p(B_{2R_0}^{(\alpha)}(t_0, x_0))} \int_0^{R_0} \left( \iint_{B_{2R_0}^{(\alpha)}(t_0, x_0)} \mu(B_r^{(\alpha)}(s, y))^{\frac{p}{p-1}} dy ds \right)^{\frac{p-1}{p}} \frac{dr}{r^{1+n}} \\
&\lesssim R_0^{2\alpha(\frac{1}{p} - \frac{1}{q})} \|G\|_{L_t^q L_x^p(\mathbb{R}_+^{1+n})} \int_0^{R_0} \left( \iint_{B_{2R_0}^{(\alpha)}(t_0, x_0)} \mu(B_r^{(\alpha)}(s, y)) dy ds \right)^{\frac{p-1}{p}} \frac{\phi(r)^{\frac{1}{p}} dr}{r^{1+n}} \\
&\lesssim R_0^{2\alpha(\frac{1}{p} - \frac{1}{q})} \|G\|_{L_t^q L_x^p(\mathbb{R}_+^{1+n})} \mu(K)^{\frac{p-1}{p}} \int_0^{R_0} \phi(r)^{\frac{1}{p}} r^{-1+2\alpha - \frac{n+2\alpha}{p}} dr.
\end{aligned}$$

Meanwhile, for  $p > q$ , it holds that

$$\begin{aligned}
& \int_0^{R_0} \iint_{B_{2R_0}^{(\alpha)}(t_0, x_0)} G(s, y) \mu(B_r^{(\alpha)}(s, y)) dy ds \frac{dr}{r^{1+n}} \\
& \leq \|G\|_{L_t^q L_x^q(B_{2R_0}^{(\alpha)}(t_0, x_0))} \int_0^{R_0} \left( \iint_{B_{2R_0}^{(\alpha)}(t_0, x_0)} \mu(B_r^{(\alpha)}(s, y))^{\frac{q}{q-1}} dy ds \right)^{\frac{q-1}{q}} \frac{dr}{r^{1+n}} \\
& \lesssim R_0^{n(\frac{1}{q} - \frac{1}{p})} \|G\|_{L_t^q L_x^p(\mathbb{R}_+^{1+n})} \int_0^{R_0} \left( \iint_{B_{2R_0}^{(\alpha)}(t_0, x_0)} \mu(B_r^{(\alpha)}(s, y)) dy ds \right)^{\frac{q-1}{q}} \frac{\phi(r)^{\frac{1}{q}} dr}{r^{1+n}} \\
& \lesssim R_0^{n(\frac{1}{p} - \frac{1}{q})} \|G\|_{L_t^q L_x^p(\mathbb{R}_+^{1+n})} \mu(K)^{\frac{q-1}{q}} \int_0^{R_0} \phi(r)^{\frac{1}{q}} r^{-1+2\alpha - \frac{n+2\alpha}{q}} dr.
\end{aligned}$$

The above estimates induce a constant  $c_0 := C(R_0, p, q, \alpha) > 0$ , depending on  $R_0$  and  $p, q, \alpha$ , such that

$$\mu(K) \lesssim c_0 \|G\|_{L_t^q L_x^p(\mathbb{R}_+^{1+n})} \left( \mu(K) + \mu(K)^{1 - \frac{1}{p \wedge q}} \int_0^{R_0} \phi(r)^{\frac{1}{p \wedge q}} r^{-1+2\alpha - \frac{n+2\alpha}{p \wedge q}} dr \right).$$

Therefore, if

$$\text{III} := \int_0^{R_0} \phi(r)^{\frac{1}{p \wedge q}} r^{-1+2\alpha - \frac{n+2\alpha}{p \wedge q}} dr < \infty,$$

then by the fact  $C_{p,q}^{(\alpha)}(K) = 0$  it follows that  $\mu(K) = 0$ , and hence  $H_{\infty}^{\phi, \alpha}(K) \approx \mu(K) = 0$ . This in turn implies  $H^{\phi, \alpha}(K) = 0$  thanks to

$$H_{\infty}^{\phi, \alpha}(\cdot) = 0 \iff H^{\phi, \alpha}(\cdot) = 0.$$

Consequently,  $H^{\phi, \alpha}(\mathcal{B}[S_{\alpha}F; p, q]) = 0$ .

The remaining is to consider two situations as follows.

*Case 1:*  $n - 2\alpha(p \wedge q - 1) > 0$ . Under this condition, we choose

$$\phi(r) := r^{\eta}, \quad \forall r \in (0, \infty) \quad \& \quad \eta > n - 2\alpha(p \wedge q - 1)$$

to obtain  $\text{III} < \infty$ , thereby reaching

$$\dim_H^{(\alpha)}(\mathcal{B}[S_{\alpha}F; p, q]) \leq n - 2\alpha(p \wedge q - 1).$$

*Case 2:*  $n - 2\alpha(p \wedge q - 1) = 0$ . Under this condition, we select

$$\phi_{\epsilon}(r) := \left( \ln_+ \frac{1}{r} \right)^{-\eta_{\epsilon}}, \quad \forall r \in (0, \infty) \quad \& \quad \eta_{\epsilon} = p \wedge q + \epsilon > p \wedge q$$

to ensure  $\text{III} < \infty$  and thus  $H^{\phi_{\epsilon}, \alpha}(\mathcal{B}[S_{\alpha}F; p, q]) = 0$ .

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